Constraints on physical amplitudes derived from a modified analytic interpolation problem

1981 J. Phys. A: Math. Gen. 14 1271

(http://iopscience.iop.org/0305-4470/14/6/007)

J. Phys. A: Math. Gen. 14 (1980) 1271–1279. Printed in Great Britain

Constraints on physical amplitudes derived from a modified analytic interpolation problem

I Caprini

Institute for Physics and Nuclear Engineering, Bucharest, POB 5206, Romania

Received 21 May 1980, in final form 26 November 1980

Abstract. Optimal constraints on the values of a physical amplitude are derived when its imaginary part and an upper bound on the modulus are given along distinct regions of the boundary of the analyticity domain. The problem is formulated rigorously as a minimum norm problem in a space of analytic functions and solved by applying a duality theorem. An approximate description of the solution, suitable for practical applications, is presented.

1. Introduction

In recent years the interpolation Schur–Carathéodory–Pick–Nevanlinna theory for analytic functions has been successfully applied to the investigation of various problems in particle theory (Nenciu 1973, Ciulli *et al* 1975, Raszillier 1979, Guiaşu *et al* 1980, Caprini and Diţă 1980). Usually in such problems one starts from the knowledge of the modulus of a physical amplitude on the boundary of the analyticity domain, using it for constraining the values of the amplitude or its derivative in interior points. In such an approach, i.e. when one constructs an analytic function starting from its modulus on the cut rather than its imaginary part, much weaker assumptions at infinity are needed, the main requirements now referring to the knowledge of the interior zeros. But the unknown factors containing the interior zeros, or more generally the inner function in the canonical factorisation (Duren 1970) of the analytic function investigated, can be optimally maximised. One obtains in this way an exact description of the domain allowed for the values of the function at any interior point, which results in rigorous bounds and correlations among coupling constants and other physical parameters of interest. However, analytic interpolation theory in its standard form, requiring the knowledge of the modulus along the whole boundary, is not suitable for some physical situations, when the imaginary (absorbtive) part of the amplitude is known accurately along a part of the boundary. In the present paper we shall formulate and solve a modified interpolation problem, suitable for these situations. In § 2, we formulate the problem and treat it by applying a duality theorem. In § 3 a convenient description of the solution, involving explicit analytic expressions, is presented.

2. Formulation of the problem

We consider the general case of a physical amplitude $f(\nu^2)$ as a function of the energy variable ν^2, and assume that $f(\nu^2)$ is real analytic in the ν^2 complex plane, cut along the real axis for $\nu^2 \geqslant \nu_0^2$. The information available on f consists of its imaginary part along the region Γ_1 of the cut, i.e.

$$\operatorname{Im} f(\nu^2+\mathrm{i}\eta)=\rho(\nu^2), \qquad \nu_0^2 \leqslant \nu^2 \leqslant \nu_{\mathrm{u}}^2, \qquad \eta \searrow 0, \tag{2.1}$$

and an upper bound on its modulus along the remaining part Γ_2 of the boundary, i.e.

$$|f(\nu^2)| \leqslant \sigma(\nu^2), \qquad \nu^2 > \nu_{\mathrm{u}}^2. \tag{2.2}$$

The properties of the boundary values $\rho(\nu^2)$ and $\sigma(\nu^2)$ will be specified below. The problem is to establish the constraining power of the input information expressed in the relations (2.1) and (2.2) upon the values of the function f or its derivatives at n points situated either inside the analyticity domain or in the region Γ_1 of the cut where the imaginary part is given. More precisely, given n points ν_k^2, $k=1, 2, \ldots, n$, which we assume for simplicity to be real and distinct, one has to find the optimal domain $\mathscr{D} \subset \mathbb{R}^n$ of the values $\{f(\nu_k^2)\}_{k=1}^n$, consistent with the conditions (2.1) and (2.2). The extension to complex points ν_k^2 and the inclusion of the derivatives of f is straightforward.

In order to solve the problem it is convenient to perform first the conformal mapping

$$z=\frac{a-(\nu_{\mathrm{u}}^2-\nu^2)^{1/2}}{a+(\nu_{\mathrm{u}}^2-\nu^2)^{1/2}}, \tag{2.3}$$

a being an arbitrary positive parameter. By this transformation the ν^2 plane is mapped onto the unit disc $|z| \leqslant 1$, such that the part of the cut where $\nu_0^2 \leqslant \nu^2 \leqslant \nu_{\mathrm{u}}^2$ becomes the real segment $[x_0, 1]$, with $x_0=[a-(\nu_{\mathrm{u}}^2-\nu_0^2)^{1/2}]/[a+(\nu_{\mathrm{u}}^2-\nu_0^2)^{1/2}]$, while the upper and lower borders of the remaining part of the cut ($\nu^2 \geqslant \nu_{\mathrm{u}}^2$) are mapped onto the upper and lower semicircles ($z=\mathrm{e}^{\mathrm{i}\theta}$) respectively. Further, we denote by z_k, $k=1, \ldots, n$, the images of the points ν_k^2, $k=1, \ldots, n$, through the transformation (2.3).

The properties of the amplitude f in the new variable z can be easily established. From (2.1) it follows that f has a cut along the segment $[x_0, 1]$, with a given discontinuity across it, equal to the function $\rho(x)$, which we assume here to be Lipschitz continuous (Duren 1970). This means that f has inside $|z| \leqslant 1$ a known non-analytic part. In what follows we shall take into account this fact by splitting the function f into two terms, one of them being analytic in $|z|<1$ and the other having in $|z|<1$ a prescribed non-analytic part. Of course, this separation is not unique but, as we shall show below, the final result will not depend on this arbitrariness. For our purposes it is convenient to write $f(z)$ as

$$f(z)=g(z)+\frac{1}{\pi}\int_{x_0}^{1+\varepsilon} \frac{\rho(x)\,\mathrm{d}x}{x-z}. \tag{2.4}$$

In this relation ε is an arbitrary positive number and $\rho(x)$ for $x>1$ is an arbitrary continuous extension of the function $\rho(x)$ known for $x \leqslant 1$. By construction, the last term in (2.4) has inside the disc $|z|<1$ a cut along the segment $[x_0, 1]$ with the same discontinuity as f. Accordingly the function $g(z)$ will be real analytic in $|z|<1$, having in particular real values for $x_0 \leqslant x \leqslant 1$.

We assume further that the boundary function σ appearing in (2.1) is such that $\ln \sigma(\theta) \in L^1[-\pi, \pi]$, and define the outer function $S(z)$ by (Duren 1970)

$$S(z)=\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z} \ln \sigma(\theta)\,\mathrm{d}\theta\right), \tag{2.5}$$

which is by construction real analytic and without zeros in $|z|<1$, having on the boundary ($z=\mathrm{e}^{\mathrm{i}\theta}$) the modulus equal to $\sigma(\theta)$. Then (2.2) can be written in the equivalent form

$$|f(\theta)/S(\theta)| \leq 1, \qquad \theta \in [-\pi, \pi].$$

By combining this relation with (2.4), we obtain the inequality

$$\left| \tilde{g}(\theta) + \frac{1}{S(\theta)\pi} \int_{x_0}^{1+\varepsilon} \frac{\rho(x)}{x - e^{i\theta}} \, dx \right| \leq 1, \qquad \theta \in [-\pi, \pi], \tag{2.6}$$

which expresses in a compact form the conditions (2.1) and (2.2) of the problem. The function $\tilde{g} = g/S$ appearing in equation (2.6) is analytic in $|z| < 1$, its values at the points z_k being related to $f(z_k)$ through

$$\tilde{g}(z_k) = \frac{1}{S(z_k)} \left(f(z_k) - \frac{1}{\pi} \int_{x_0}^{1+\varepsilon} \frac{\rho(x)\, dx}{x - z_k} \right) \tag{2.7}$$

if z_k is below the threshold x_0 and

$$\tilde{g}(z_k) = \frac{1}{S(z_k)} \left(\operatorname{Re} f(z_k) - \frac{P}{\pi} \int_{x_0}^{1+\varepsilon} \frac{\rho(x)\, dx}{x - z_k} \right) \tag{2.7'}$$

if z_k is in the interval $[x_0, 1]$. The principal part appearing in front of the last integral exists, due to the assumed smoothness properties of $\rho(x)$. By the above manipulations, the initial problem of finding the admissible values of $f(z_k)$ was reduced to the equivalent problem of describing the optimal domain of values $\tilde{g}(z_k)$, when the analytic function $\tilde{g}$ is subject to the unique condition (2.6). By means of the relations (2.7) and (2.7′), this domain will then be expressed in terms of the values $f(z_k)$ of interest.

We start by noticing that the values $\{\tilde{g}(z_k)\}_{k=1}^{n}$ consistent with (2.6) form a closed and convex domain $\mathscr{D}$ in the real Euclidean space $\mathbb{R}^n$. Let us take first a point inside this domain, having the coordinates $\tilde{g}(z_k)^{(i)}$, $k = 1, \ldots, n$. This means that one can find at least one analytic function $\tilde{g}$ which takes at the points z_k the prescribed values $\tilde{g}(z_k)^{(i)}$ and satisfies the inequality (2.6). Therefore if one calculates the L^∞ norm, i.e. the essential supremum with respect to $\theta \in [-\pi, \pi]$ of the left-hand side of (2.6), for a fixed $\tilde{g}$ and then takes the infimum of the numbers thus obtained with respect to all the functions $\tilde{g}$ analytic in $|z| < 1$, having the values $\tilde{g}(z_k)$ equal to $\tilde{g}(z_k)^{(i)}$, the result will be surely less than one. On the other hand, if a point with coordinates $\tilde{g}(z_k)^{(e)}$ is outside $\mathscr{D}$, the L^∞ norm of the function appearing in (2.6) will be strictly greater than one for all the analytic functions $\tilde{g}$ assuming at z_k the given values $\tilde{g}(z_k)^{(e)}$. If one takes again the infimum of all such L^∞ norms and takes into account the fact that it is effectively achieved by some analytic function $\tilde{g}$ (Adamyan *et al* 1968, Duren 1970), one will obtain also a number strictly greater than one. From the above arguments, it follows that the domain $\mathscr{D}$ of the admissible values $\tilde{g}(z_k)$ is exactly described by the inequality

$$\min_{\substack{\tilde{g} \in H^\infty \\ \tilde{g}(z_k) = \text{given}}} \left\| \tilde{g}(\theta) + \frac{1}{\pi S(\theta)} \int_{x_0}^{1+\varepsilon} \frac{\rho(x)\, dx}{x - e^{i\theta}} \right\|_\infty \leq 1, \tag{2.8}$$

which is saturated, as follows from convexity arguments, if the point $\{\tilde{g}(z_k)\}_{k=1}^{n}$ belongs to the frontier of $\mathscr{D}$.

In the above relation we have restricted the minimisation to the analytic functions $\tilde{g}$ of class H^∞ (Duren 1970), i.e. bounded in $|z| \leq 1$. Actually, this restriction did not appear in our previous discussion; however, since the last term appearing in (2.8) is by construction bounded on the boundary of the unit disc, it is enough to consider in the minimisation process only functions $\tilde{g}$ bounded in $|z|\leq 1$, the L^∞ norm being otherwise infinite and hence of no interest to us.

We were thus led to an L^∞ norm minimisation in the H^∞ Banach space, with constraints at interior points (Krein and Nudelman 1973, Adamyan *et al* 1968, Duren 1970). We shall treat it by applying the standard duality theorem for minimum norm problems (Duren 1970). In the next section a convenient approximate description of the solution will be given.

As the points $\{z_k\}_{k=1}^n$ were assumed for simplicity to be real and distinct, we can write $\tilde{g}(z)$ as

$$\tilde{g}(z)=\sum_{k=1}^{n} A_k B_k(z)+h(z)B_{n+1}(z). \tag{2.9}$$

In this relation the functions $B_k(z)$ are products of Blaschke factors defined recurrently as

$$B_1(z)=1, \qquad B_k(z)=B_{k-1}(z)\frac{z-z_{k-1}}{1-zz_{k-1}}, \qquad k=2,\ldots,n+1, \tag{2.10}$$

and the coefficients A_k are determined in terms of $\tilde{g}(z_k)$ from the triangular system of equations

$$\tilde{g}(z_k)=\sum_{j=1}^{k} A_j B_j(z_k), \qquad k=1,\ldots,n. \tag{2.11}$$

The function $h(z)$ is an analytic function, arbitrary in H^∞, since the constraints upon $g(z)$ are automatically fulfilled by the expression (2.9). Writing the inequality (2.8) in terms of h, we obtain

$$\min_{h\in H^\infty}\left\|h(\theta)+\sum_{k=1}^{n} A_k\frac{B_k(\theta)}{B_{n+1}(\theta)}-\frac{1}{\pi S(\theta)B_{n+1}(\theta)}\int_{x_0}^{1+\varepsilon}\frac{\rho(x)\,\mathrm{d}x}{\mathrm{e}^{\mathrm{i}\theta}-x}\right\|_\infty\leq 1, \tag{2.12}$$

using the property that the Blaschke factors (2.10) have modulus equal to one on the boundary. We now apply the duality theorem (Duren 1970) relating a minimum norm problem in H^∞ to a supremum problem in the unit sphere S^1 of the Hardy Banach space H^1. The relation (2.12) is therefore equivalent to

$$\sup_{\substack{F\in H^1\\ \|F\|_1\leq 1}}\left|\frac{1}{2\pi\mathrm{i}}\int_{|z|=1}F(z)\left(\sum_{k=1}^{n} A_k\frac{B_k(z)}{B_{n+1}(z)}-\frac{1}{\pi S(z)B_{n+1}(z)}\int_{x_0}^{1+\varepsilon}\frac{\rho(x)\,\mathrm{d}x}{z-x}\right)\mathrm{d}z\right|\leq 1, \tag{2.13}$$

where we have denoted by $\|F\|_1$ the L^1 norm of F on the boundary. The integral appearing in (2.13) can be computed exactly by applying the residue theorem. By taking into account the poles produced by the Blaschke factors, one finds that the first term is equal to

$$\sum_{k=1}^{n} A_k \sum_{j=k}^{n} F(z_j)\left(\frac{(z-z_j)B_k(z)}{B_{n+1}(z)}\right)_{z=z_j}.$$

If one now permutes the sums upon k and j, i.e. $\Sigma_{k=1}^n\,\Sigma_{j=k}^n=\Sigma_{j=1}^n\,\Sigma_{k=1}^j$, and recalls the equations (2.11), one obtains finally the contribution of the first term in (2.13) in the form

$$\frac{1}{2\pi\mathrm{i}}\int_{|z|=1}F(z)\sum_{k=1}^{n} A_k\frac{B_k(z)}{B_{n+1}(z)}\,\mathrm{d}z=\sum_{k=1}^{n} F(z_k)\tilde{g}(z_k)\left(\frac{z-z_k}{B_{n+1}(z)}\right)_{z=z_k}. \tag{2.14}$$

For the evaluation of the second term in the integral (2.13) we apply again the residue theorem, taking into account the fact that the integrand now has a cut along the segment $[x_0, 1]$. First we write this term as

$$\frac{1}{2\pi\mathrm{i}}\int_{|z|=1}\frac{F(z)}{\pi S(z)B_{n+1}(z)}\left(\int_{x_0}^{1+\varepsilon}\frac{\rho(x)}{z-x}\,\mathrm{d}x\right)\mathrm{d}z$$
$$=+\sum_{z_k<x_0}\frac{F(z_k)}{\pi S(z_k)}\left(\frac{z-z_k}{B_{n+1}(z)}\right)_{z=z_k}\times\int_{x_0}^{1+\varepsilon}\frac{\rho(x)\,\mathrm{d}x}{z_k-x}-\frac{1}{2\pi\mathrm{i}}\int_C\frac{F(z)}{\pi S(z)B_{n+1}(z)}\left(\int_{x_0}^{1+\varepsilon}\frac{\rho(x)}{z-x}\,\mathrm{d}x\right)\mathrm{d}z, \tag{2.15}$$

where the sum contains the contribution of the poles z_k situated below the threshold x_0, while C is a contour around the real segment $[x_0, 1]$. The explicit evaluation of this contour integral gives

$$\frac{1}{2\pi\mathrm{i}}\int_C\mathrm{d}z\frac{F(z)}{\pi S(z)B_{n+1}(z)}\int_{x_0}^{1+\varepsilon}\frac{\rho(x)}{z-x}\,\mathrm{d}x$$
$$=\frac{1}{2\pi\mathrm{i}}\left(\int_{x_0}^{1}\mathrm{d}y\frac{F(y)}{\pi S(y)B_{n+1}(y+\mathrm{i}\eta)}\int_{x_0}^{1+\varepsilon}\frac{\rho(x)}{y-x+\mathrm{i}\eta}\,\mathrm{d}x-\int_{x_0}^{1}\frac{F(y)}{\pi S(y)B_{n+1}(y-\mathrm{i}\eta)}\int_{x_0}^{1+\varepsilon}\frac{\rho(x)\,\mathrm{d}x}{y-x-\mathrm{i}\eta}\right). \tag{2.16}$$

By using the identity

$$\int_{x_0}^{1+\varepsilon}\frac{\rho(x)}{x-y\pm\mathrm{i}\eta}\,\mathrm{d}x=\mathrm{P}\int_{x_0}^{1+\varepsilon}\frac{\rho(x)}{x-y}\,\mathrm{d}x\mp\mathrm{i}\pi\rho(y)$$

and applying again the residue theorem for the poles of the Blaschke factors situated inside the contour C (i.e. for $z_k > x_0$), one writes (2.16) in the form

$$I=-\sum_{z_k>x_0}\frac{F(z_k)}{\pi S(z_k)}\left(\frac{z-z_k}{B_{n+1}(z)}\right)_{z=z_k}\mathrm{P}\int_{x_0}^{1+\varepsilon}\frac{\rho(x)\,\mathrm{d}x}{z_k-x}-\frac{\mathrm{P}}{\pi}\int_{x_0}^{1}\frac{F(x)\rho(x)}{B_{n+1}(x)S(x)}\,\mathrm{d}x. \tag{2.17}$$

If the relations (2.14), (2.15) and (2.17) are introduced in (2.13) and the equalities (2.7) and (2.7′) are taken into account, it may be seen that the intermediate values $\tilde{g}(z_k)$ as well as the arbitrary function $\rho(x)$ for $x>1$ disappear from the result, which is expressed only in terms of the input values $f(z_k)$, $k=1,\ldots,n$, $\rho(x)$ for $x_0\leqslant x\leqslant 1$ and $\sigma(\theta)$ (contained in $S(z)$), as

$$\sup_{\substack{F\in H^1\\ \|F\|_1\leqslant 1}}\left|\sum_{z_k<x_0}\frac{F(z_k)f(z_k)}{S(z_k)}\left(\frac{z-z_k}{B_{n+1}(z)}\right)_{z=z_k}+\sum_{z_k>x_0}\frac{F(z_k)\,\mathrm{Re}\,f(z_k)}{S(z_k)}\left(\frac{z-z_k}{B_{n+1}(z)}\right)_{z=z_k}-\frac{\mathrm{P}}{\pi}\int_{x_0}^{1}\frac{F(x)\rho(x)}{S(x)B_{n+1}(x)}\,\mathrm{d}x\right|\leqslant 1. \tag{2.18}$$

We remark that the same result would be obtained if one started with a more general decomposition of f in two terms than that given in (2.4). For instance, if one writes

$$f(z)=g(z)+\frac{1}{\pi\varphi(z)}\int_{x_0}^{1+\varepsilon}\frac{\rho(x)\varphi(x)}{x-z}\,\mathrm{d}x,$$

where $\varphi(z)$ is analytic without zeros in $|z|<1$ and repeats the above calculations, one finds that $\varphi(z)$ does not appear in the final result (2.18). For the evaluation of the supremum (2.18) it is useful to apply a factorisation theorem (Duren 1970), which expresses every function $F(z)$ from the unit sphere of H^1 as a product of two functions from the unit sphere of the Hilbert space H^2. In particular, let us start from the canonical factorisation (Duren 1970) of $F(z)$, writing it as

$$F(z)=F^{(i)}(z)F^{(o)}(z) \tag{2.19}$$

where $F^{(i)}$ and $F^{(o)}$ are the inner and outer factors of F respectively (Duren 1970). Let us define further

$$w(z)=[F^{(o)}(z)]^{1/2}, \qquad G(z)=F^{(i)}(z)[F^{(o)}(z)]^{1/2}. \tag{2.20}$$

It may be easily verified that both w and G are functions from the unit sphere of H^2 ($w(z)$ is actually an outer function) and

$$F(z)=w(z)G(z). \tag{2.21}$$

Using this factorisation in (2.18), one obtains

$$\sup_{\substack{w\in H^2,\, G\in H^2\\ \|w\|_2\leq 1,\, \|G\|_2\leq 1}} \left|\sum_{k=1}^{n} \frac{w(z_k)G(z_k)}{S(z_k)} \operatorname{Re} f(z_k)\left(\frac{z-z_k}{B_{n+1}(z)}\right)_{z=z_k} - \frac{\mathrm{P}}{\pi}\int_{x_0}^{1} \frac{w(x)G(x)\rho(x)}{S(x)B_{n+1}(x)}\,\mathrm{d}x\right| \leq 1, \tag{2.22}$$

where for simplicity we have written in a compact form the contribution of the points $z_k<x_0$ and $z_k>x_0$, the notation $\operatorname{Re} f(z_k)$ being in the first case redundant, since the values $f(z_k)$, for $z_k<x_0$, are real.

Using the expansions

$$w(z)=\sum_{j=0}^{\infty} w_j z^j, \qquad G(z)=\sum_{j=0}^{\infty} G_j z^j, \tag{2.23}$$

the supremum (2.22) can be written in the form

$$\sup_{\substack{\sum_{j=0}^{\infty} w_j^2\leq 1\\ \sum_{j=0}^{\infty} G_j^2\leq 1}} \left|\sum_{i,j=1}^{\infty} H_{ij} w_{i-1} G_{j-1}\right| \leq 1, \tag{2.24}$$

where H is an infinite Hankel matrix defined as

$$H_{ij}=c_{-(i+j-1)}, \qquad i,j=1,2,\ldots, \tag{2.25}$$

in terms of the real numbers

$$c_{-(j+1)}=\sum_{k=1}^{n} z_k^j \frac{\operatorname{Re} f(z_k)}{S(z_k)}\left(\frac{z-z_k}{B_{n+1}(z)}\right)_{z=z_k} - \frac{\mathrm{P}}{\pi}\int_{x_0}^{1} \frac{x^j\rho(x)\,\mathrm{d}x}{B_{n+1}(x)S(x)}, \qquad j=0,1,2,\ldots, \tag{2.26}$$

which are actually the negative-frequency Fourier coefficients of the function multiplying $F(z)$ in the integral (2.13).

The relation (2.24) expresses the result of the problem in terms of the norm of a Hankel matrix. Actually, it is known that for a general infinite matrix the norm is not always attained in l^2 (Adamyan *et al* 1968). For the practical calculation of such a norm, it is convenient to truncate the matrix H at a finite range, by setting all the coefficients $c_{-(j+1)}=0$ for $j>N$. The norm of a finite matrix $H^{(N)}$ is known to be equal to the square root of the greatest eigenvalue of $H^{(N)+}H^{(N)}$ and can be computed using standard numerical methods (for a detailed description of such a computational program see Caprini *et al* (1979)). It can be shown (Adamyan *et al* 1968) that when $N \to \infty$ the norms of $H^{(N)}$ approach the exact norm of H, i.e. the supremum (2.24). In this way the problem of constraining the values $f(z_k)$ of the amplitude f by means of the initial conditions (2.1) and (2.2) is completely solved. The parameters $f(z_k)$ are contained implicitly in the norm of the matrix H, the inequality (2.24) yielding therefore the exact domain $\mathscr{D}$ allowed for these values. However, the above form of the solution is not always very convenient in practical applications. In the next section an approximate description of $\mathscr{D}$, very suitable for practical applications, will be presented.

3. Approximate description of the solution

We remark that the complicated form of the solution found in the previous section was due to the fact that we had to treat a minimum norm problem in the Banach space H^∞. Usually such a problem is much simpler if it is formulated in a Hilbert space, such as H^2 (Duren 1970). In order to exploit this idea, let us consider again the relation (2.22) and assume that the maximisation there is performed only with respect to the function $G \in H^2$, the outer function w being kept fixed. It is evident that, for a fixed w, one obtains in this way a supremum lower than the true one. Accordingly, if this new supremum is bounded by one, a constraint upon the values $f(z_k)$, weaker than the exact one (2.24), will be obtained. If one denotes by $\mathscr{D}_w$ the domain of $f(z_k)$ obtained in this way, it is clear that $\mathscr{D} \subset \mathscr{D}_w$ for every fixed $w \in H^2$ with $\|w\|_2 \leqslant 1$, and therefore $\mathscr{D} \subset \bigcap_{w \in H^2, \|w\|_2 \leqslant 1} \mathscr{D}_w$. Moreover, since taking the supremum over w in (2.22) is equivalent to finding the optimal domain $\mathscr{D}$, one can see that actually

$$\mathscr{D} = \bigcap_{\substack{w \in H^2 \\ \|w\|_2 \leqslant 1}} \mathscr{D}_w. \tag{3.1}$$

The important point of the present approach is that for every fixed w the domains $\mathscr{D}_w$ can be explicitly described in a simple form. Moreover, as we shall discuss below, with a convenient choice of w one expects to approximate the exact domain $\mathscr{D}$ very closely by $\mathscr{D}_w$. From (2.22) it follows that the domain $\mathscr{D}_w$ is yielded by the inequality

$$\sup_{\sum_{j=0}^{\infty} G_j^2 \leqslant 1} \left| \sum_{j=0}^{\infty} G_j \gamma_{-(j+1)} \right| \leqslant 1 \tag{3.2}$$

where G_j are the Fourier coefficients of $G(z)$ from (2.23) and the real numbers $\gamma_{-(j+1)}$ are

$$\gamma_{-(j+1)} = \sum_{k=1}^{n} z_k^j \frac{\operatorname{Re} f(z_k) w(z_k)}{S(z_k)} \left(\frac{z - z_k}{B_{n+1}(z)} \right)_{z=z_k}$$
$$- \frac{\mathrm{P}}{\pi} \int_{x_0}^{1} \frac{\rho(x) w(x) x^j}{B_{n+1}(x) S(x)} \, \mathrm{d}x, \qquad j = 0, 1, \ldots . \tag{3.3}$$

The supremum (3.2) is evaluated immediately by applying the Cauchy–Schwarz theorem, yielding the inequality

$$\sum_{j=0}^{\infty} \gamma_{-(j+1)}^2 \leqslant 1. \tag{3.4}$$

By introducing the expression (3.3) of $\gamma_{-(j+1)}$ in this relation and noticing that the infinite summation over j can be exactly performed (actually the order of integration and infinite summation can be interchanged, due to the smoothness properties assumed for $\rho(x)$), one obtains the following inequality:

$$\sum_{i,k=1}^{n} \frac{\operatorname{Re} f(z_i) \operatorname{Re} f(z_k)}{1-z_i z_k} \frac{w(z_i)w(z_k)}{S(z_i)S(z_k)} \left(\frac{z-z_k}{B_{n+1}(z)}\right)_{z=z_k} \left(\frac{z-z_i}{B_{n+1}(z)}\right)_{z=z_i}$$
$$-2\sum_{k=1}^{n} \operatorname{Re} f(z_k) \frac{w(z_k)}{S(z_k)} \left(\frac{z-z_k}{B_{n+1}(z)}\right)_{z=z_k} \frac{\mathrm{P}}{\pi} \int_{x_0}^{1} \frac{\rho(x)w(x)\,\mathrm{d}x}{S(x)B_{n+1}(x)(1-xz_k)}$$
$$+\frac{\mathrm{P}}{\pi^2} \int_{x_0}^{1} \int_{x_0}^{1} \frac{\rho(x)\rho(y)w(x)w(y)\,\mathrm{d}x\,\mathrm{d}y}{S(x)S(y)B_{n+1}(x)B_{n+1}(y)(1-xy)} \leqslant 1. \tag{3.5}$$

This relation yields the required explicit description of the domain $\mathscr{D}_w$ of the values $f(z_k)$. Stated otherwise, it represents a set of necessary conditions which must be satisfied by these values, for all the functions $w \in H^2$, with $\|w\|_2 \leqslant 1$. Of course, one is interested in finding among these conditions the optimal one, which amounts to performing the additional optimisation upon w, which remained from (2.22). The above approach is useful in practice only if one succeeds in approximating quite closely the exact domain $\mathscr{D}$ by using a convenient choice of w, instead of performing the whole maximisation which would lead again to the complicated solution (2.24).

Such a useful choice of w is suggested by the following remark: one can easily see that the domain $\mathscr{D}_w$ is yielded in an equivalent way by the inequality

$$\min_{\substack{\tilde{g} \in H^\infty \\ \tilde{g}(z_k) = \text{given}}} \left\| w(\theta)\left(\tilde{g}(\theta) + \frac{1}{\pi S(\theta)} \int_{x_0}^{1+\varepsilon} \frac{\rho(x)\,\mathrm{d}x}{x-\mathrm{e}^{\mathrm{i}\theta}}\right)\right\|_2 \leqslant 1, \tag{3.6}$$

where w is, as before, a fixed function in H^2 with $\|w\|_2 \leqslant 1$. Indeed, this follows immediately by applying to (3.6) the same arguments as those leading from (2.8) to (2.22), the L^∞ norm being now replaced by the L^2 norm, and taking into account the fact that by duality a minimum norm problem in H^2 is related to a maximisation in the unit sphere of H^2. Since the L^2 norm weighted by a function w with $\|w\|_2 \leqslant 1$ as in (3.6) is always less than the L^∞ norm (2.8), it is again evident that $\mathscr{D}_w$ is larger than $\mathscr{D}$ and contains it. On the other hand, it is known that the L^∞ norm of an arbitrary function f given on the boundary of the unit disc, $z = \mathrm{e}^{\mathrm{i}\theta}$, is equal to the supremum of the L^2 norms of f computed in all the variables $z' = \mathrm{e}^{\mathrm{i}\theta'}$ obtained from z by a conformal mapping of the unit disc onto itself, $z' = (z-\alpha)/(1-\alpha^* z)$. Let us therefore take (Caprini and Diţă 1980)

$$w(z) = (1-\alpha^2)^{1/2}/(1-\alpha z), \tag{3.7}$$

where α is a parameter (real, for our purposes), $|\alpha| < 1$, such that $|w(\theta)|^2 = (1-\alpha^2)/(1+\alpha^2-2\alpha\cos\theta)$ is the Jacobian of the above conformal mapping from z to z'. By using (3.7) in the inequality (3.5), with α variable in the range $(-1, 1)$, which is equivalent to solving the L^2 norm minimisation for an arbitrary deformation of the frontier of the unit disc, one expects to obtain a good approximation, from outside, of the exact domain $\mathscr{D}$. Since the domains $\mathscr{D}_w$ are explicitly given by the simple expression (3.7), this procedure of approximating $\mathscr{D}$ is very economical and convenient for practical applications.

References

Adamyan V M, Arov D Z and Krein M G 1968 *Funkcional'nyi Analiz i Ego Prilojeniya* **2** 1 (Engl. transl. 1968 *Functional Analysis and its Applications* **2** 1)
Caprini I and Diţă P 1980 *J. Phys. A: Math. Gen.* **13** 1265
Caprini I, Săraru M, Pomponiu C, Ciulli M, Ciulli S and Stefănescu I S 1979 *Comput. Phys. Commun.* **18** 305
Ciulli S, Pomponiu C and Sabba-Stefănescu I 1975 *Phys. Rep.* **17** 133
Duren P 1970 *Theory of H^p-Spaces* (New York: Academic)
Guiaşu I, Radescu E and Raszillier I 1980 *Ann. Phys.* **127** 436
Krein M G and Nudelman A A 1973 *The problem of Markov moments and extremal problems* (Moscow: Nauka) (in Russian)
Nenciu G 1973 *Nucl. Phys.* B **53** 584
Raszillier I 1979 *Rev. Roum. Phys.* **24** 873